Title: Using the Additive Property of Compactly Supported Cohomology Groups

Author: Małgorzata Aneta Marciniak

Comments: LaTeX, 6 pages

Subj-class: Several Complex Variables

MCS-class: 32C36 (primary); 16E99 (secondary)

Abstract: This article uses basic homological methods for evaluating examples of compactly supported cohomology groups of line bundles over projective curve.

1 Introduction

Compactly supported cohomology groups play a key role in complex analysis. Vanishing properties are related to solvability of $\overline{\partial}$ *problem as well as analytic continuation problems ([4] and [5]). This article uses the Künneth formula and the additive property for finding examples of compactly supported cohomology groups. The additive property requires the definition of the inverse image of a sheaf, which is shorty explained in subsection 1.3. Compactly supported cohomology groups of* $\mathbb{C}^1$, $\mathbb{C}^*$, $\mathbb{C}^2 \setminus \{(0,0)\}$ *and* E_{-1} *are evaluated in subsection 2.1 and of* E_k*, the line bundles over* $\mathbb{P}^1$*, in subsection 2.2. An example of a trivial bundle over* $\mathbb{P}^1$ *can use he Künneth formula in place of the additive property, is presented in subsection 2.3.*

1.1 Definitions and Properties

Definition 1.1 **(Compactly supported Dolbeault cohomology groups)** *Compactly supported Dolbeaut cohomology groups of the domain* D *are the complex vector spaces:*

$$\mathfrak{H}_c^{p,q}(D) = \frac{\{\overline{\partial}\text{-closed forms with compact support of bidegree } (p,q) \text{ in } D\}}{\{\overline{\partial}\text{-exact forms with compact support of bidegree } (p,q) \text{ in } D\}}.$$

The following theorem shows the relationship between compactly supported cohomology groups and compactly supported Dolbeault cohomology groups.

Theorem 1.1 **(Dolbeault's Theorem, [3])** *If* D *is an open domain in the space of* n *complex variables,* $\mathscr{O}$ *is the sheaf of germs of holomorphic functions on* D*, and* $\mathfrak{H}_c^{p,q}(D)$ *is compactly supported Dolbeault cohomology group of bidegree* (p,q) *for* D*. Then* $H_c^q(D,\mathscr{O}) = \mathfrak{H}_c^{0,q}(D)$*.* ▮

An alternate definition can be found in [2]. Note that if X *is a compact manifold then* $H_c^n(X,\mathscr{O}) = H^n(X,\mathscr{O})$ *fr all* $n = 0,1,2,\ldots$*.*

In particular, $H_c^0(X,\mathscr{O}) = \mathfrak{H}_c^{0,0}(X)$ *denotes (a linear space of) global holomorphic functions on* X *with compact support. Note that*

$$H_c^0(X,\mathscr{O}) = \begin{cases} 0 & \text{if } X \text{ is noncompact,} \\ \mathbb{C} & \text{if } X \text{ is compact.} \end{cases}$$

In particular, $H_c^1(X,\mathscr{O}) = \mathfrak{H}_c^{0,1}(X)$ *can be seen as follows, where* ω *has a bidegree* $(0,1)$*:*

$$H_c^1(X,\mathscr{O}) = \frac{\{\omega \text{ with compact support such that } \overline{\partial}\omega = 0\}}{\{\omega \text{ with compact support such that } \overline{\partial} f = \omega \text{ for some } f \text{ with compact support}\}}$$

1.2 The Künneth Formula

The following exact sequences are obtained for each n *separately.*

Theorem 1.2 *If X and Y are locally compact Hausdorff spaces, with the sheaves $\mathscr{F}$ and $\mathscr{G}$ respectively and $\mathscr{F} * \mathscr{G} = 0$, then the sequence*

$$0 \to \bigoplus_{p+q=n} H_c^p(X,\mathscr{F}) \otimes H_c^q(Y,\mathscr{G}) \to H_c^n(X \times Y, \mathscr{F} \otimes \mathscr{G}) \to \bigoplus_{p+q=n+1} H_c^p(X,\mathscr{F}) * H_c^q(Y,\mathscr{G}) \to 0$$

is exact.

Here $*$ denotes the free product. Note if one of the factors is torsion free then the free product is 0.

1.3 The additive property

Compactly supported cohomology groups have all properties of a cohomology theory. The "additive" property, broadly used in the further part of the research, requires the notion of the inverse image of a sheaf.

Definition 1.2 (Inverse image) *Let $f : A \to B$ be a map and let $\mathscr{G}$ be a sheaf on B with canonical projection $\pi : \mathscr{G} \to B$. The inverse image sheaf $f^*\mathscr{G}$ is defined as*

$$f^*\mathscr{G} = \{(a, g) \in A \times \mathscr{G} : f(a) = \pi(g)\}.$$

In particular, if f is a closed embedding, the following theorem holds.

Theorem 1.3 ([10] III.7.6) *Let $i : Y \to X$ be a closed embedding, then the following sequence*

$$\ldots \to H_c^q(X \setminus Y, \mathscr{F}) \to H_c^q(X, \mathscr{F}) \to H_c^q(Y, i^*\mathscr{F}) \to H_c^{q+1}(X \setminus Y, \mathscr{F}) \to \ldots,$$

is exact. ▮

2 Examples for the Additive Property

2.1 Removing a point

Let us find compactly supported cohomology groups of $\mathbb{C}^1$ using the representation $\mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}$

Example 2.1 *Since $\mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}$, then $X = \mathbb{P}^1$, $Y = \infty$ and $X \setminus Y = \mathbb{C}^1$ implies the following exact sequence:*

$$\begin{gathered} 0 \to H_c^0(\mathbb{C}^1, \mathscr{O}) \to H_c^0(\mathbb{P}^1, \mathscr{O}) \to H_c^0(\{\infty\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^1, \mathscr{O}) \to \\ H_c^1(\mathbb{P}^1, \mathscr{O}) \to H_c^1(\{\infty\}, i^*\mathscr{O}) \to 0. \end{gathered} \tag{1}$$

Since $H_c^0(\mathbb{C}^1, \mathscr{O}) = 0$, $H_c^1(\{\infty\}, i^*\mathscr{O}) = 0$, and $H_c^0(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, the exact sequence can be written as follows:

$$0 \to \mathbb{C} \to H_c^0(\{\infty\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^1, \mathscr{O}) \to 0.$$

We need to find $H_c^0(\{\infty\}, i^*\mathscr{O})$. In particular, the sheaf $i^*\mathscr{O}$ is simply $\mathscr{O}_X/\mathscr{O}_{X\setminus Y}$, since the point $Y = \{\infty\}$ is closed in X. The global functions at ∞ with coefficients in $\mathscr{O}_{\mathbb{P}^1}/\mathscr{O}_{\mathbb{C}^1}$ are those convergent series at ∞, which are 0 in a neighborhood of ∞, so they are the germs of holomorphic functions of one variable at ∞. The exact sequence gives $H_c^1(\mathbb{C}^1, \mathscr{O}) = H_c^0(\{\infty\}, i^*\mathscr{O})/\mathbb{C}$ in this sense, that two germs f and g represent distinct elements of the group if $f(\infty) \neq g(\infty)$. In other words:

$$H_c^1(\mathbb{C}^1, \mathscr{O}) = \{\sum_{i<0} a_i z^i, a_i \in \mathbb{C}\},$$

where the series converges at infinity.

A similar procedure can be applied to $H_c^1(\mathbb{C}^*, \mathscr{O})$.

Example 2.2 *Note that* $\mathbb{P}^1 = \mathbb{C}^* \cup \{\infty\} \cup \{0\}$. *We could equivalently use* $\mathbb{C}^1 = \mathbb{C}^* \cup \{0\}$ *and relate to the previous result. The additive property provides the following exact sequence:*

$$0 \to H_c^0(\mathbb{C}^*, \mathscr{O}) \to H_c^0(\mathbb{P}^1, \mathscr{O}) \to H_c^0(\{0, \infty\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^*, \mathscr{O}) \to H_c^1(\mathbb{P}^1, \mathscr{O}) \to H_c^1(\{0, \infty\}, i^*\mathscr{O}) \to 0.$$

After applying $H_c^0(\mathbb{C}^*, \mathscr{O}) = H_c^1(\mathbb{P}^1, \mathscr{O}) = H_c^1(\{0, \infty\}, i^*\mathscr{O}) = 0$ *and* $H_c^0(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, *the exact sequence simplifies to:*

$$0 \to \mathbb{C} \to H_c^0(\{0, \infty\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^*, \mathscr{O}) \to 0.$$

Since $H_c^0(\{0, \infty\}, i^*\mathscr{O})/\mathbb{C}$ *consist, we have*

$$H_c^1(\mathbb{C}^*, \mathscr{O}) = \{\sum_{i<0} a_i z^i, a_i \in \mathbb{C}\} \oplus \{\sum_{i>0} b_j w^j, b_j \in \mathbb{C}\},$$

where the series converge in a neighborhood of ∞ *and* 0, *respectively.*

Example 2.3 *The group* $H_c^*(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O})$ *can be found from the additive property. Let* $X = \mathbb{C}^2$ *and* $Y = \{(0,0)\}$, *then*

$$0 \to H_c^0(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O}) \to H_c^0(\mathbb{C}^2, \mathscr{O}) \to H_c^0(\{(0,0)\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O}) \to$$

$$H_c^1(\mathbb{C}^2, \mathscr{O}) \to H_c^1(\{(0,0)\}, i^*\mathscr{O}) \to H_c^2(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O}) \to H_c^2(\mathbb{C}^2, \mathscr{O}) \to 0.$$

The only nontrivial groups remain:

$$0 \to H_c^0(\{(0,0)\}, i^*\mathscr{O}) \to H_c^1(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O}) \to 0$$

thus

$$H_c^1(\mathbb{C}^2 \setminus \{(0,0)\}, \mathscr{O}) = H_c^0(\{(0,0)\}, i^*\mathscr{O}) = \{\sum_{i,j>0} a_{ij} z^i w^j, a_{ij} \in \mathbb{C}\},$$

where the series converges in some neighborhood of $(0,0)$.

We will start with a basic example of a line bundle over $\mathbb{P}^1$.

Example 2.4 *Using the fact that* $E_{-1} = \mathbb{P}^2 \setminus \{p\}$, *we can evaluate compactly supported cohomology groups of* E_{-1}. *The additive property gives the exact sequence:*

$$0 \to H_c^0(E_{-1}, \mathscr{O}) \to H_c^0(\mathbb{P}^2, \mathscr{O}) \to H_c^0(\{p\}, i^*\mathscr{O}) \to H_c^1(E_{-1}, \mathscr{O}) \to H_c^1(\mathbb{P}^2, \mathscr{O}) \to$$

$$H_c^1(\{p\}, i^*\mathscr{O}) \to H_c^2(E_{-1}, \mathscr{O}) \to H_c^2(\mathbb{P}^2, \mathscr{O}) \to 0.$$

Since $\mathbb{P}^2$ *is compact* $H_c^1(\mathbb{P}^2, \mathscr{O}) = H_c^2(\mathbb{P}^2, \mathscr{O}) = 0$ *and* $H_c^0(\mathbb{P}^2, \mathscr{O}) = \mathbb{C}$. *Then the sequences converts to:*

$$0 \to H_c^0(\mathbb{P}^2, \mathscr{O}) \to H_c^0(\{p\}, i^*\mathscr{O}) \to H_c^1(E_{-1}, \mathscr{O}) \to 0$$

and

$$0 \to H_c^1(\{p\}, i^*\mathscr{O}) \to H_c^2(E_{-1}, \mathscr{O}) \to 0.$$

Since $H_c^1(\{p\}, i^*\mathscr{O}) = 0$ *because of dimensional reasons, we obtain that* $H_c^2(E_{-1}, \mathscr{O}) = 0$. *The preceding exact sequence proves that* $H_c^1(E_{-1}, \mathscr{O}) = H_c^0(\{p\}, i^*\mathscr{O})/\mathbb{C}$, *which in the terms of convergent series can be written as:*

$$H_c^1(E_{-1}, \mathscr{O}) = \{\sum_{(n,m)>(0,0)} a_{n,m} z^n w^m, a_{n,m} \in \mathbb{C}\},$$

where the series converges near $(0,0)$ *in the local coordinates.*

2.2 Removing a Projective Curve

The total space of the line bundle E_k with $k \in \mathbb{Z}$ consists of two coordinate patches $X_1 \simeq \mathbb{C}^2$ and $X_2 \simeq \mathbb{C}^2$ with coordinates (z_1, w_1) and (z_2, w_2) respectively, related on $X_1 \cap X_2$ according to the rule $z_1 = \frac{1}{z_2}$ and $w_1 = z_2^k w_2$. It is a well known fact that $H^1_c(E_k, \mathscr{O}) = 0$ for $k > 0$, nevertheless, we will present how to obtain this result using the additive property.

In the previous section we found compactly supported cohomology groups of E_{-1} using the representation $E_{-1} = \mathbb{P}^2 \setminus \{p\}$, since E_{-1} can be obtained by removing a point from the projective plane. This is not true for other line bundles over $\mathbb{P}^1$. We will consider the Hirzebruch surfaces $\mathscr{H}_k$ and the representation $E_{-k} = \mathscr{H}_k \backslash \mathbb{P}^1$. The Hirzebruch surface $\mathscr{H}_k$ consists of four coordinate patches $X_0, X_1, X_2, X_3, \simeq \mathbb{C}^2$ with $(z_j, w_j) \in X_j$ and the transition functions described below:

$$
\begin{aligned}
z_1 &= \frac{1}{z_0}, & w_1 &= z_0^k w_0 \\
z_2 &= z_1, & w_2 &= \frac{1}{w_1} \\
z_3 &= \frac{1}{z_2}, & w_3 &= z_2^{-k} w_2 \\
z_3 &= z_0, & w_3 &= \frac{1}{w_0}.
\end{aligned}
$$

Let $Y_j \simeq \mathbb{P}^1$ and let us denote $i_j : Y_j \to \mathscr{H}_k$ with the following order:

$$
\begin{aligned}
\mathbb{C}^1 \times \mathbb{P}^1 &= \mathscr{H}_k \setminus Y_0 \\
E_{-k} &= \mathscr{H}_k \setminus Y_1 \\
\mathbb{C}^1 \times \mathbb{P}^1 &= \mathscr{H}_k \setminus Y_2 \\
E_k &= \mathscr{H}_k \setminus Y_3
\end{aligned}
$$

Here Y_0 is a a projective curve in $X_0 \cup X_3$ described by equations $z_0 = 0$ and $z_3 = 0$. If $f_0(z_3, w_3) \in i_0^* \mathscr{O}$ then on $X_0 \cap X_3$:

$$f_0(z_3, w_3) = \sum_{(n,m) \geq (0,0)} a_{n,m} z_3^n w_3^m = \sum_{(n,m) \geq (0,0)} a_{n,m} z_0^n \frac{1}{w_0^m},$$

which proves that $m = 0$ so f does not depend on w_3 and

$$f_0(z_3, w_3) = \sum_{n \geq 0} a_n z_3^n.$$

Thus $H^0_c(Y_0, i_0^* \mathscr{O}) = \{\sum_{n \geq 0} a_n z_3^n, an \in \mathbb{C}\}$, where the series converges in a neighborhood of $z_0 = 0$. The following exact sequence

$$0 \to H^0_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) \to H^0_c(\mathscr{H}_k, \mathscr{O}) \to H^0_c(Y_0, i_0^* \mathscr{O}) \to H^1_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) \to H^1_c(\mathscr{H}_k, \mathscr{O}) \to$$

$$H^1_c(Y_0, i_0^* \mathscr{O}) \to H^2_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) \to H^2_c(\mathscr{H}_k, \mathscr{O}) \to 0$$

can be simplified to the sequences:

$$0 \to H^0_c(\mathscr{H}_k, \mathscr{O}) \to H^0_c(Y_0, i_0^* \mathscr{O}) \to H^1_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) \to 0$$

and

$$0 \to H^2_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) \to H^2_c(\mathscr{H}_k, \mathscr{O}) \to 0.$$

Then $H^1_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) = \{\sum_{n>0} a_n z_3^n,\, a_n \in \mathbb{C}\}$ and $H^2_c(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O}) = 0$.

Note that Y_1 is a projective curve in $X_0 \cup X_1$ described in the local coordinates as $w_0 = 0$ and $w_1 = 0$. If $f_1(z_1, w_1) \in i_1^*\mathscr{O}$ then on $X_0 \cap X_1$:

$$f_1(z_1, w_1) = \sum_{(n,m)\geq(0,0)} a_{n,m} z_1^n w_1^m = \sum_{(n,m)\geq(0,0)} a_{n,m} \frac{1}{z_0^n} \left(z_0^k w_0\right)^m = \sum_{(n,m)\geq(0,0)} a_{n,m} z_0^{(km-n)} w_0^m,$$

which shows that $km - n \geq 0$. Thus $H_c^0(Y_1, i_1^*\mathscr{O}) = \{ \sum_{(n,m)\geq(0,0)} a_{n,m} z_1^n w_1^m : km - n \geq 0,\, a_{n,m} \in \mathbb{C}\}$. The following exact sequence

$$0 \to H_c^0(E_{-k}, \mathscr{O}) \to H_c^0(\mathscr{H}_k, \mathscr{O}) \to H_c^0(Y_1, i_1^*\mathscr{O}) \to H_c^1(E_{-k}, \mathscr{O}) \to H_c^1(\mathscr{H}_k, \mathscr{O}) \to$$

$$H_c^1(Y_1, i_1^*\mathscr{O}) \to H_c^2(E_{-k}, \mathscr{O}) \to H_c^2(\mathscr{H}_k, \mathscr{O}) \to 0$$

can be simplified to the sequences:

$$0 \to H_c^0(\mathscr{H}_k, \mathscr{O}) \to H_c^0(Y_1, i_1^*\mathscr{O}) \to H_c^1(E_{-k}, \mathscr{O}) \to 0$$

and

$$0 \to H_c^2(E_{-k}, \mathscr{O}) \to H_c^2(\mathscr{H}_k, \mathscr{O}) \to 0.$$

Thus $H_c^1(E_{-k}, \mathscr{O}) = \{ \sum_{(n,m)>(0,0)} a_{n,m} z_1^n w_1^m : km - n \geq 0,\, a_{n,m} \in \mathbb{C}\}$ and $H_c^2(E_{-k}, \mathscr{O}) = 0$.

Computations for Y_2 (that is a submanifold in $X_1 \cup X_2$) are similar to those for Y_0 and overall $H_c^0(Y_2, i_2^*\mathscr{O}) = \{\sum_{n\geq 0} a_n z_1^n,\ {}_n \in \mathbb{C}\}$, where the series converges in a neighborhood of $z_1 = 0$.

Note that Y_3 is a projective curve in $X_2 \cup X_3$ described in local coordinates by $w_2 = 0$ and $w_3 = 0$. If $f_3(z_3, w_3) \in i_3^*\mathscr{O}$ then on $X_2 \cap X_3$:

$$f_3(z_3, w_3) = \sum_{(n,m)\geq(0,0)} a_{n,m} z_3^n w_3^m = \sum_{(n,m)\geq(0,0)} a_{n,m} \frac{1}{z_2^n} \left(z_2^{-k} w_2\right)^m = \sum_{(n,m)\geq(0,0)} a_{n,m} z_0^{(-km-n)} w_0^m,$$

which shows that $-km - n \geq 0$ that is possible only for $m = n = 0$. Thus $H_c^0(Y_2, i_2^*\mathscr{O}) = \mathbb{C}$.

The following exact sequence

$$0 \to H_c^0(E_k, \mathscr{O}) \to H_c^0(\mathscr{H}_k, \mathscr{O}) \to H_c^0(Y_3, i_3^*\mathscr{O}) \to H_c^1(E_k, \mathscr{O}) \to H_c^1(\mathscr{H}_k, \mathscr{O}) \to$$

$$H_c^1(Y_3, i_3^*\mathscr{O}) \to H_c^2(E_k, \mathscr{O}) \to H_c^2(\mathscr{H}_k, \mathscr{O}) \to 0$$

can be simplified to the following:

$$0 \to H_c^0(\mathscr{H}_k, \mathscr{O}) \to H_c^0(Y_3, i_3^*\mathscr{O}) \to H_c^1(E_k, \mathscr{O}) \to 0$$

and

$$0 \to H_c^2(E_k, \mathscr{O}) \to H_c^2(\mathscr{H}_k, \mathscr{O}) \to 0.$$

Then $H_c^1(E_k, \mathscr{O}) = \mathbb{C}/\mathbb{C} = 0$ and $H_c^2(E_k, \mathscr{O}) = 0$.

2.3 Example for the Künneth Formula

This section contains an example of a surface $\mathbb{P}^1 \times \mathbb{C}^1$.

Example 2.5 *Let us find $H_c^i(\mathbb{P}^1 \times \mathbb{C}^1, \mathscr{O})$ for $i = 0, 1, 2$ using the Künneth formula for products. Recall the following groups $H_c^0(\mathbb{P}^1, \mathscr{O}) = \mathbb{C}$, $H_c^1(\mathbb{P}^1, \mathscr{O}) = 0$, $H_c^0(\mathbb{C}^1, \mathscr{O}) = 0$ and $H_c^1(\mathbb{C}^1, \mathscr{O}) = \{\sum_{s \geq 0} a_s z^s, a_s \in \mathbb{C}\}$. Then the Künneth Formula for $\mathbb{P}^1 \times \mathbb{C}^1$ gives the following for the first cohomology group of $\mathbb{P}^1 \times \mathbb{C}^1$:*

$$0 \to \bigoplus_{p+q=1} H_c^p(\mathbb{C}^1, \mathscr{O}) \otimes H_c^q(\mathbb{P}^1, \mathscr{O}) \to H_c^1(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O}) \to \bigoplus_{p+q=2} H_c^p(\mathbb{C}^1, \mathscr{O}) * H_c^q(\mathbb{P}^1, \mathscr{O}) \to 0,$$

which converts to:

$$0 \to H_c^1(\mathbb{C}^1, \mathscr{O}) \bigoplus H_c^0(\mathbb{P}^1, \mathscr{O}) \to H_c^1(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O}) \to H_c^1(\mathbb{C}^1, \mathscr{O}) * H_c^1(\mathbb{P}^1, \mathscr{O}) \to 0,$$

and gives $H_c^1(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O}) = H_c^1(\mathbb{C}^1, \mathscr{O}) = \{\sum_{s \geq 0} a_s z^s$ and the series converges in a neighborhood of $0\}$. Similarly for $H_c^2(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O})$:

$$0 \to H_c^1(\mathbb{C}^1, \mathscr{O}) \otimes H_c^1(\mathbb{P}^1, \mathscr{O}) \to H_c^2(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O}) \to 0,$$

which gives $H_c^2(\mathbb{C}^1 \times \mathbb{P}^1, \mathscr{O}) = 0$.

References

[1] L. Bonavero and M. Brion (Eds.), *Geometry of toric varieties. Lectures from the Summer School held in Grenoble, June 19–July 7, 2000. Séminaires et Congres, 6. Société Mathématique de France, Paris (2002).*

[2] G. E. Bredon, *Sheaf Theory, 2 ed., Springer, (1997).*

[3] P. Dolbeault, *Sur la cohomologie des variétés analytiques complexes C. R. Acad. Sci. Paris, 236 (1953) 175 - 177.*

[4] R. Dwilewicz, *Additive Riemann-Hilbert problem in line bundles over $\mathbb{CP}^1$, Canadian Math. Bull.* **49** *(2006), 72 - 81.*

[5] R. Dwilewicz, *Holomorphic extensions in complex fiber bundles, J. Math. Analysis and Appl.* **322** *(2006),*

[6] R. Dwilewicz, *An analytic point of view at toric varieties, Serdica Math. J.* **33** *(2007), 163 - 240.*

[7] B. Gilligan, A.T. Huckleberry, *Complex homogeneous manifolds with two ends, Michigan Math. J. (1981).*

[8] H. Freudenthal, *Über die Enden topologischer Räume und Gruppen, Math. Z.* **33** *(1931), 692-713.*

[9] W. Fulton, *Introduction to Toric Varieties, Princeton Univ. Press, NJ, (1993).*

[10] B. Iversen, *Cohomology of Sheaves, Springer-Verlag, (1986).*

[11] S.G. Krantz, *Function Theory of Several Complex Variables, AMS Chelsea Publishing, (2001).*

[12] M. A. Marciniak, *Holomorphic Extensions in Toric Surfaces, Journal of Geometric Analysis 2011, 1-23.*